Finite element model of a beam structure with piezoelectric patches using RL shunt circuits

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Abstract

The system studied in this paper consists of a beam structure, with two non-collocated piezoelectric patches attached to its surface. One patch produces the mechanical displacement of the beam by applying an electrical voltage, while the other converts this mechanical displacement into electrical energy which is then dissipated through passive electrical networks. By using the notion of neutral axis for this asymmetric system and Hamilton principle, a finite element model, incorporating the electrical dynamics of the connected passive electrical circuits is presented. Series and parallel RL circuits are considered here. Basing on the finite element model, an iterative method to determine optimum values of the RL circuits is studied; performance comparison between series and parallel RL circuits for vibration suppression is made.

Keywords: Hamilton principle, neutral axis, finite elements model, non-collocated piezoelectric patches, RL shunted circuits, vibration suppression.

1. Introduction

The beam structures containing piezoelectric materials are used in many applications such as alignment of optical fibers [1], piezoelectric miniature robots [2], deformable mirror [3], micropumps [4], valves [5], ultrasonic motors [6] and vibration suppression [7]. To optimize these systems and to explore new applications, their numerical modeling is required. The aim of our study is to develop a finite element model to modulate a beam structure with non-collocated piezoelectric patches bonded on its surface where it is few studied in literature compared to collocated piezoelectric patches bonded on a beam structure [8-13] and then to use this model for vibration damping. Vibration damping using piezoelectric elements is also very investigated in literature [14-20]. Passive vibration damping using RL shunt circuit is chosen to study in this paper. The system studied consists in an aluminum cantilever beam actuated by a PZT patch, whose flexural vibration is transmitted to a non-collocated PZT sensor, which is connected to a series or parallel RL shunt circuits for damping the vibration (fig.1). The main point of our paper is to develop a finite element model for a beam structure with two non-collocated piezoelectric patches bonded on it and to incorporate the electrical dynamics of the connected shunt circuits. By an easy iterative way, scanning R and L between two boundaries, we can obtain the optimal values of R and L in series and parallel connections. Performance comparison between series and parallel RL circuits for vibration suppression is made after obtaining the RL optimal circuit. At the end of this study, conclusion and perspectives are presented.

2. Constitutive equations

The assumption of linear elastic constitutive equation of materials allows us writing the stress ten-
sor as a function of the strain tensor and vice versa.

\[
\begin{align*}
\{\sigma\} &= [c] \{\varepsilon\} \\
\{\varepsilon\} &= [s] \{\sigma\}
\end{align*}
\] (1)

Where \(\{\sigma\}\) is the stress tensor, \(\{\varepsilon\}\) is the strain tensor, \([s]\) is the elastic stiffness tensor and \([c]\) is the compliance tensor.

Under the assumption of linear piezoelectric constitutive equations we can write

\[
\begin{align*}
\{\sigma\} &= \{[c]^{\text{E}}\} \{\varepsilon\} - \{\varepsilon\} \{[e]^{\text{E}}\} \\
\{[D]\} &= \{[e]\} \{\varepsilon\} + \{[e]^{\text{E}}\} \{[E]\}
\end{align*}
\] (2)

Where \([e]\) is the piezoelectric coefficient tensor, \([e]^{\text{E}}\) is the dielectric permittivity at constant strain condition and \([c]^{\text{E}}\) is the elastic stiffness tensor at constant electric field.

### 3. Model hypothesis

We define a displacement field vector by the displacement of a point \(M(x,y,z)\) of the system to \(M'\) at instant \(t\).

\[
\text{MM}'(x,y,z,t) = \{u\} = (u_i(x,y,z,t))
\] (3)

Where \(u_i\) is the displacement along \(i\)-axis. We consider this displacement field as the unknown must be determined to know the status of the system at all times.

Equations of the system are derived based on the following assumptions:

1. Euler-Bernoulli hypothesis i.e. uniaxial stress in the \(x\)-direction, small deformation and cross section remain perpendicular to the neutral axis after deformation.
2. The electric field is assumed to be uniformly distributed in the \(z\)-direction \(\{E\} = E_0(x,t)\)
3. The poling direction of the piezoelectric layer is in the positive \(z\)-direction
4. Perfect bonding between the piezoelectric layer and aluminum layer
5. No external forces are applied to the system

Then, the displacement field becomes

\[
\{u\} = \begin{cases} 
 u_1(x,y,z,t) & \approx -(z-z_n) \frac{\partial}{\partial z} w \\
 u_2(x,y,z,t) & = 0 \\
 u_3(x,y,z,t) & \approx w(x,t)
\end{cases}
\] (4)

Where \(w(x,t)\) is the transverse displacement of the neutral axis of the system \(z_n\). \(z_n\) can be calculated by setting to zero the sum of all forces in \(x\)-direction over the entire cross-section [21], here \(z_n\) is computed from the bottom of the system according to the coordinate system taken in figure 1:

\[
b \int_0^{l_m} \sigma_m^m(x) \, dx + b \int_0^{l_m} \sigma_p^p(x) \, dx = 0
\] (5)

Where \(\sigma_m^m\) and \(\sigma_p^p\) are referred to as stresses induced in the piezoelectric and aluminum layers in \(x\)-direction. According to uniaxial stress assumption, we can write

\[
\sigma_1^m = c_m \varepsilon_1,
\]

\[
\sigma_2^p = c_p \varepsilon_1 - e_p E_3 = \frac{1}{s_{13}} \varepsilon_1 - \frac{d_{31}}{s_{11}} E_3,
\]

\[
D_3 = c_p \varepsilon_1 + e_p E_3 = \frac{d_{31}}{s_{11}} \varepsilon_1 + \varepsilon_3 \frac{d_{33}}{s_{11}} E_3
\] (6)

Where \([e]^{\text{E}} = c_p = \frac{1}{s_{11}}\) is the Young modulus for the piezoelectric layer, \([e] = e_p = \frac{d_{31}}{s_{11}}\) is the piezoelectric constant, \([c] = c_m\) is the Young modulus for the aluminum layer, \(s_{11}^{\text{E}}\) is the elastic compliance for the piezoelectric layer at constant electric field, \(d_{31}\) is the piezoelectric constant and \([e]^{\text{E}} = e_p^{\text{E}} = e_{33}^{\text{E}} - \frac{d_{33}}{s_{11}}\) where \(e_{33}^{\text{E}}\) is the electrical permittivity at constant stress for the piezoelectric layer. Properties and geometric parameters for the PZT and aluminum layers are given in Table 1. \(\varepsilon_1\) is the strain in the \(x\)-direction and is defined by

\[
\{\varepsilon\} = \varepsilon_1 = \frac{\partial}{\partial x} u_1 = -(z-z_n) \frac{\partial^2}{\partial x^2} w
\] (7)

Utilizing Hook’s law, while substituting strain relationship from (5), yields:

\[
j_0 \int_0^{l_m} c_m (z-z_n) \, dz + j_0 \int_0^{l_m} \varepsilon_3 \frac{d_{33}}{s_{11}} E_3 \, dz = 0
\] (8)

Upon simplifying, the neutral axis \(z_n\) can be obtained as shown in equation (9) at the end of the paper.

### 4. Variational equation

Hamilton principle is the most widely used one for dynamic structures [7]. All variations at each known instant are zero.

\[
\delta \int_{t_1}^{t_2} (L + W) \, dt = 0
\] (10)

Where \(L\) is the Lagrangian \((L = J - H\) with \(J\) the kinetic energy and \(H\) the global enthalpy) and
W is the virtual work of external mechanical and

table

<table>
<thead>
<tr>
<th>Properties/materials</th>
<th>PZT</th>
<th>Aluminum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus (Pa)</td>
<td>$c_n = 69 \times 10^9$</td>
<td></td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu_m = 0.33$</td>
<td></td>
</tr>
<tr>
<td>Volume density (Kg.m$^3$)</td>
<td>$\rho_p = 7900$</td>
<td>$\rho_m = 2700$</td>
</tr>
<tr>
<td>Relative permittivity</td>
<td>$\varepsilon_{rP} = 1282.1$</td>
<td></td>
</tr>
<tr>
<td>Piezoelectric constant (m.V$^{-1}$)</td>
<td>$d_{11} = -1.3e^{11}$</td>
<td></td>
</tr>
<tr>
<td>Elastic compliances (Pa$^{-1}$)</td>
<td>$s_{fi} = 1.3e^{11}$</td>
<td></td>
</tr>
<tr>
<td>Length x width x thickness (mm$^3$)</td>
<td>32x17x0.27</td>
<td>180x17x0.5</td>
</tr>
</tbody>
</table>

Table 1: properties and geometric parameters for the PZT and aluminum layers

electric forces. The kinetic energy is equal to ($\rho$ is the volume density)

$$J = \int_V \frac{1}{2} \rho [\dot{u}]^T [\dot{u}] dV$$  \(11\)

The global enthalpy is equal to

$$H = \int_V \frac{1}{2} [\sigma]^T [\sigma] - \{B\}^T \{D\} dV$$  \(12\)

And finally, the virtual work of external mechanical and electrical forces is equal to

$$W = \int_V [\dot{u}]^T \{F_v\} dV + t_p [E] Q$$  \(13\)

Where $\{F_v\}$ represents the body applied forces, $t_p$ the thickness of piezoelectric layer, $Q$ the electric charge.

By replacing $[\sigma]$ and $[D]$ by their equation (2) in (12), integrating by parts the kinetic energy (11) and applying Hamilton principle, the variational equation is obtained in (14).

$$\{\delta u\} = \left\{ \begin{array}{c} -\left( z - z_n \right) \partial_x \delta w \\ 0 \\ \delta w(x, t) \end{array} \right\}$$  \(15\)

$$\{\delta e\} = \delta e_1 = -\left( z - z_n \right) \partial_x^2 \delta w$$  \(16\)

Integrating equation (14) over the entire volume of the system, we obtain the equation (17).

After incorporating the electrical circuit in the model, three electrical impedances (R, RL series & RL parallel) were connected to the piezoelectric patch p2. These kinds of circuits are known in the literature by the single mode damping [22]
because they are able to suppress the vibration for one single frequency i.e. for each frequency there are an optimal electric circuit. We take here, the fourth resonant mode of vibration at frequency \( f_0 = 484.3 \) Hz to determine optimal values of the electrical circuits. The process stills the same for any other frequency.

Let us beginning with the first electrical circuit where a resistance is connected to \( p_2 \). By scanning the resistance and calculating the displacement using the finite element model for each resistance, we can find the optimal value of resistance corresponding to a minimum displacement of the beam. Figure 2, shows the optimal resistance obtained using our finite element model and it is equal to 15 K\( \Omega \). It is noted in simulation that, for any resonant frequency taken, the optimal resistance value is approximately equal to the value obtained in the formula of equation (28).

\[
R_{opt} = \frac{1}{2\pi f_c C_p}
\]  
\( (28) \)

Where \( f_c \) is the resonance frequency and \( C_p \) is the electrical capacitance of the PZT sensor and can be calculated using table 2.

<table>
<thead>
<tr>
<th>( C_p )</th>
<th>Simulation</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{K_{np2}}{t_p^2} )</td>
<td>20</td>
<td>( \frac{\varepsilon_{33}^2 f_p b^2}{t_p} )</td>
</tr>
<tr>
<td>( C_p(nF) )</td>
<td>22</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Comparison between simulation and theory for the capacitance value obtained

The graph at the top of figure 2 shows the tip displacement of the beam function of resistance at the fourth resonant frequency and the graph at the bottom shows displacement versus frequency around the fourth resonant one, for three different values of resistance. Figure 2, can be shown that the damping suppression is not very sensitive for resistance shunt circuit.

In case of RL shunt circuit connected to \( p_2 \), there are two parameters (R & L) to be varied to get the minimum displacement of the beam. Let us starting by fixing the resistance to the optimal value obtained by resistance shunt circuit and, varied the inductance to get the optimal value. Figure 3, shows the optimal inductance at fixed R=15 \( R \) for parallel and series connections. The obtained value of L is equal to the inductance tuning the shunt effect to the frequency of the disturbance, according to (29).

\[
L_{opt} = \frac{1}{(2\pi f_c)^2 C_p}
\]  
\( (29) \)

It is noted in simulation that, for any resonant frequency taken, the optimal inductance is always verifying equation (29).

According to equation (26), in series connection, for small values of R compared to \( jwL \), the impedance behaves like an inductor and, for large values of R compared to \( jwL \), the influence of L is not considered. In parallel connection, for small values of R compared to \( jwL \), L is not considered and for large values of R compared to \( jwL \), L dominates. We can notice by simulation that, for any R values between these two extremes, the optimal inductance is still verify equation (29) and it is independent of R values in series and parallel connections. Wu [23] demonstrated that for series RL shunt, the optimal inductance is dependant of R values in case of large R. As mentioned before, there is no influence of L to the
system displacement in case of large $R$ values and, the electric circuit can be considered as a resistance shunt. Being given the optimal inductance $5.3 \, H$ verifying equation (29), scanning $R$ values to find the optimal one. Figure 4 & 5; show the tip displacement versus frequency for different $R$ values. Figure 4, shows that, for large $R$ values the tip displacement is approximately equal to that obtained in figure 2, when resistance shunt circuit is used. And for small $R$ values, the system behavior corresponds to an inductance only. Less than 4 $K\Omega$, the system behavior becomes slightly inductance. Inversely in the case of parallel connection, the tip displacement is approximately equal to that obtained in figure 2, for small $R$ values and inductance behavior for large $R$ values. 75 $K\Omega$ can be chosen as an optimal $R$ value.

Rebuild figure 3 at optimal $R$ values as shown in figure 6. It was expected to obtain the same optimal inductance with a smaller displacement. Figure 7 shows displacement comparison between open circuit, optimal resistance shunt circuit, optimal RL shunt circuit in both series and parallel connections. The vibration suppression performance is evaluated based on amplitude of the displacement. The best performance is obtained for minimal displacement. According to figure 7, the optimal RL parallel shunt circuit shows a performance slightly better than the RL series shunt circuit.

7. Conclusion

Our paper is the first one in the literature that studied the damping vibration for non-collocated piezoelectric patches bonded on a thin beam structure in the configuration presented in figure 1. In this paper we developed a finite element model of the system and then, we incorporate the electrical dynamics of the connected passive electrical circuits. Optimal values for the shunt circuits are determined by an iterative method using the finite element model.

The model studied, including the passive electrical network must be verified experimentally in future work and this new configuration of system.
used for damping application must be compared in term of performance with the other existing one. The finite element model developed here will be taken as support to study a non-collocated piezoelectric patches bonded on a plate, for the same type of applications.

\[
\begin{aligned}
\frac{1}{2} c_m t_m^2 + c_p t_p^2 + 2 c_p t_p t_m & \quad \text{if } x_{p1} < x < x_{p1} + l_p \text{ and } x_{p2} < x < x_{p2} + l_p \\
\frac{t_m}{2} & \quad \text{if not}
\end{aligned}
\]

(9)

\[
\int_{\Omega} \left( -\rho \left\{ \delta u \right\}^T \left\{ \delta \dot{u} \right\} [1] \{\varepsilon\} [1] \{E\} + [\delta E]^T \{\varepsilon\} [1] \{E\} + [\delta E]^T \{\varepsilon\} [1] \{E\} + [\delta u]^T \{K_1\} \right) dV + \\
t_p \{\delta E\} Q = 0
\]

(14)

\[
\int_{\Omega}^{p1}( A(x, t) dx + \int_{\partial \Omega}^{p1} \left[ B(x, t) + q e_p E_{p1}(t) \partial^2_x \delta w + \delta E_{p1}(t) q e_p \partial^2_x \delta w \right] \{E\} dx + \\
\int_{\partial \Omega}^{p2}( A(x, t) dx + \int_{\partial \Omega}^{p2} \left[ B(x, t) + q e_p E_{p2}(t) \delta^2_x \delta w + \delta E_{p2}(t) q e_p \partial^2_x \delta w \right] \{E\} dx + \\
\int_{\partial \Omega}^{p1+p2} A(x, t) dx + \int_{\partial \Omega}^{p1+p2} \left[ B(x, t) + q e_p E_{p1+p2}(t) \partial^2_x \delta w + \delta E_{p1+p2}(t) q e_p \partial^2_x \delta w \right] \{E\} dx + \\
\int_{\partial \Omega}^{p1+p2} A(x, t) dx - \\
\int_{\partial \Omega}^{p1} \partial^2_x \delta w + \delta E_{p1}(t) \partial^2_x \delta w + \delta E_{p2}(t) \partial^2_x \delta w + \delta E_{p1+p2}(t) \partial^2_x \delta w - \int_{\partial \Omega}^{p1+p2} \delta \dot{u} \{F\} = 0
\]

(17)

Where

\[
A(x, t) = \rho_m l_m \partial_x \delta w. \partial^2_x \delta w \quad + b \rho_m t_m \partial_x \delta w. \partial^2_x \delta w \quad + l_m \varepsilon_m \partial_x \delta w. \partial^2_x \delta w
\]

(18)

\[
B(x, t) = (p_p t_p + \rho_m l_m) \partial_x \delta w. \partial^2_x \delta w \quad + b (p_p t_p + \rho_m t_m) \delta w. \partial^2_x \delta w \quad + (l_p t_p + l_m c_m) \partial^2_x \delta w. \partial^2_x \delta w
\]

(19)

\[
l_m = b \int_0^{l_m} (z - z_n)^2 dz = b \frac{1}{3} [(t_m - z_n)^3 + z_n^3]
\]

(20)

\[
l_p = b \int_0^{l_p} (z - z_n)^2 dz = b \frac{1}{3} [(t_p + t_m - z_n)^3 - (t_m - z_n)^3]
\]

(21)

\[
q = b \int_0^{l_p} (z - z_n) dz = b \frac{1}{2} [(t_p + t_m - z_n)^2 - (t_m - z_n)^2]
\]

(22)

\[
[M 0 0] [\dot{U}_i 1 \dot{E}_{p1}] + [K_{mm} \quad K_{mvp1} \quad K_{mvp2}] [U_i 1 \dot{E}_{p1}] = [F_i 1 \dot{t}_P Q_{p1}]
\]

(23)

\[
[M 0 0] [\dot{U}_i 1 \dot{E}_{p3}] + [C 0 0] \begin{bmatrix} \dot{U}_i 1 \dot{E}_{p3} \\ \dot{E}_{p3} \end{bmatrix} + [K_{mm} \quad K_{mvp1} \quad K_{mvp2}] [U_i 1 \dot{E}_{p3}] = [F_i 1 \dot{t}_P Q_{p2}]
\]

(24)

\[
[M 0 0] \begin{bmatrix} \dot{U}_i 1 \dot{E}_{p1} \dot{E}_{p3} \end{bmatrix} = \begin{bmatrix} [C 0 0] & [K_{mm} \quad K_{mvp1} \quad K_{mvp2}] & [U_i 1 \dot{E}_{p1} \dot{E}_{p3}] & = [F_i 1 \dot{t}_P Q_{p1} \dot{t}_P Q_{p2}]
\]

(27)
References


