Discontinuous Galerkin Method for Computing Induced Fields in Superconducting Materials

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A discontinuous Galerkin method is proposed for computing the current density in superconductors characterized by a constitutive power law between the current density and the electric field. The method is formulated to solve the nonlinear diffusion problem satisfied by the electric field, both in the scalar and 2-D vectorial case. Application examples are given for a superconducting cylinder subjected to an external magnetic field. Results are compared to those given by the mixed finite-element/finite-volume method and those obtained using a standard finite-element software. Efficiency and robustness of the approach are illustrated on an example where the exponent in the power law is spatially dependent.

***Index Terms*—Discontinuous Galerkin method, finite-element method, finite-volume method, nonlinear diffusion, superconductors.**

I. Introduction

THE constitutive power laws $E-J$ and $J-E$ are widely used to characterize high temperature superconductors. They are written as

$$\frac{\vec{J}}{J_c} = \left\| \frac{\vec{E}}{E_c} \right\|^{\frac{1}{n}-1} \frac{\vec{E}}{E_c} \tag{1}$$

$$\frac{\vec{E}}{E_c} = \left\| \frac{\vec{J}}{J_c} \right\|^{n-1} \frac{\vec{J}}{J_c} \tag{2}$$

where $\vec{J}$ is the current density, $\vec{E}$ the electric field, E_c the critical electric field, J_c the critical current density, and n the power law exponent. The case $n=1$ corresponds to a normal conductor, while $n=+\infty$ represents the critical state model suggested by Bean [1]. Several numerical methods have been proposed to solve nonlinear diffusion problems resulting from Maxwell's equations [2]–[5]. Their results are satisfying when n is uniform and sufficiently small. Few of them are suited when n is large, and models where n locally varies are uncommon.

In this paper, we present a Discontinuous Galerkin (DG) method for computing induced fields in superconductors. We work on solving the nonlinear diffusion problem in terms of the electric field in order to determine the current density when n is large or locally varies. DG methods are well suited to treat discontinuous forms and use a piecewise high-order polynomials basis for reducing spurious oscillations. In addition, they are naturally well adapted for parallel computing.

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II. The Differential System

In a two-dimensional setting where the magnetic induction depends only on two space variables ($\vec{B}=(B_x,B_y)$), the electric field $\vec{E}$ and current density $\vec{J}$ have a single nonzero component and can thus be treated as scalar fields E and J. We set $u=E/E_c$, $\beta(u)=u^{1/n}=J/J_c$, and $c=\mu_0 J_C/E_c$. The superconductor sample has a vacuum magnetic permeability μ_0. Denoting the superconductor domain by Ω and its border by $\partial\Omega$, Maxwell's equations and the constitutive law (1) lead to the following nonlinear diffusion problem:

$$(S)\begin{cases} \frac{\partial\beta(u)}{\partial t} - \frac{1}{c}\triangle u = 0 \text{ in } \Omega \\ \vec{\nabla}u\cdot\vec{\nu} = C_b(t) \text{ on } \partial\Omega. \end{cases} \tag{3}$$

The system is established with a zero initial condition and the boundary condition on $\partial\Omega$ results from Faraday's law:

$$C_b(t) = E_c^{-1}\left(\frac{\partial B_y}{\partial t}, -\frac{\partial B_x}{\partial t}\right)\cdot\vec{\nu} \tag{4}$$

where $\vec{\nu}$ is the outward normal vector.

III. Discontinuous Galerkin Method

Let us consider a triangulation $\mathcal{I}_h = \bigcup K$ of the domain Ω. The DG approach combines discretization tools of finite-element (FE) and finite-volume (FV) methods. It consists in solving on each K the weak formulation of the system (3):

$$c\int_K \frac{\partial\beta(u)}{\partial t}\varphi dK - \int_K \vec{\nabla}u\cdot\vec{\nabla}\varphi dK - \int_\Gamma \vec{\nabla}u\cdot\vec{\nu}\varphi d\Gamma = 0 \tag{5}$$

where φ is a test function and $\Gamma=\partial K$ is an interface between two elements of $\mathcal{I}_h$ or a part of $\partial\Omega$.

A. FE Discretization on Each Element

On each triangle $K\in\mathcal{I}_h$ a finite-element approximation space is defined. Its basis functions are Lagrange polynomials of degree p. The number of nodes on K is given by $(1/2)(p+1)(p+2)$. The discrete solution is written as

$$u^K = \sum_j u_j^K \varphi_j^K, \tag{6}$$

with u_j^K its value at node j. Since β is a Liptschitz function, we assume that

$$\beta(u^K) = \sum_j \beta\left(u_j^K\right)\varphi_j^K. \tag{7}$$

The elementary mass matrix M^K is determined from the L^2 scalar product:

$$M_{ij}^K = \int_K \varphi_i^K \varphi_j^K dK \tag{8}$$

and the global mass matrix M is block diagonal.

B. Flux Term on the Interface Between Two Elements

As in an FV method, the interface term $\int_\Gamma \vec{\nabla} u \cdot \vec{\nu} \varphi d\Gamma$ is treated by a numerical flux F, which verifies $F^{K,L} = -F^{L,K}$, where K and L are neighboring elements. Its construction needs the following functions at the interface $K \cap L$: the mean value

$$\{u\} = \frac{u^L + u^K}{2} \tag{9}$$

and the jump

$$[|u|] = u^L - u^K. \tag{10}$$

Many expressions of F have been proposed in the case of the Laplace operator. We choose the expression of F based on the Non-symmetric Interior Penalty method (NIP). The NIP method consists of introducing a penalty term $\int_\Gamma \theta [|u|][|\varphi|] d\Gamma$, in order to guarantee continuity of u and $\vec{\nabla} u$ at the interface [6]. The numerical flux is given by

$$F \rightarrow \int_\Gamma \{\vec{\nabla} u\} \cdot [|\varphi|]\, \vec{\nu} d\Gamma + \int_\Gamma \theta\, [|u|]\, [|\varphi|]\, d\Gamma \tag{11}$$

where θ is a positive parameter.

C. The Discrete Problem

Rules for evaluating the different terms of the weak formulation (5) exploit properties of mesh parametrization also called "mapping". The mapping is based on a bijective function Ψ such as $\Psi(x,y) = (\xi,\eta)$, which allows to transform the physical space (x,y) into a (reference) parametric space (ξ,η) [7] (see Fig. 1). The basis functions in the parametric space are the linear combinations of $\xi^\alpha \eta^\beta$, where $\alpha+\beta \le p$. The terms of the weak formulation (5) are evaluated in the parametric space, where derivation and integration operations are more convenient, and then mapped in the physical space.

After time discretization a discrete problem is obtained on each triangle K:

$$M^K \frac{\beta\left(u_{k+1}^K\right) - \beta\left(u_k^K\right)}{\delta t} = f^K\left(u_{k+1}^K\right) \tag{12}$$

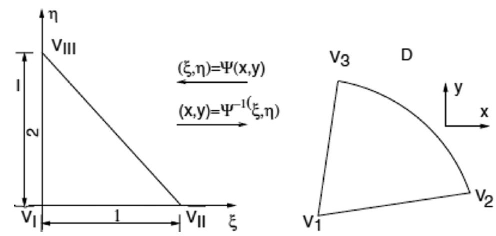

Fig. 1. Mapping of a triangle, from the (reference) parametric space to the physical space.

where δt is the time step, u_k^K is the solution at instant t_k, and $f^K(u_{k+1}^K)$ represents the discretization of the Laplace operator with numerical flux given in (11).

Unfortunately $\beta'(0) \rightarrow +\infty$ and we cannot directly use a Newton iterative method to solve (12). We thus use the change of variables: $v = \beta(u)$ and $g^K(v) = f^K(\beta^{-1}(v))$, so that the discrete problem (12) becomes

$$M^K \frac{v_{k+1}^K - v_k^K}{\delta t} = g^K\left(v_{k+1}^K\right) \tag{13}$$

which can be solved using a Newton iterative method since g^K is continuous and differentiable.

Considering the contributions of all the elements in the mesh (dropping the K superscript), we introduce the global unknown vector $dv = v_{k+1} - v_k$ which will be evaluated at each iteration (l) of the Newton method, and we define h such as

$$h(dv) = \delta t g(v_k + dv) - M dv = 0. \tag{14}$$

The iterative solution $dv^{(l+1)}$ is obtained using the following relation:

$$\frac{\partial h\left(dv^{(l)}\right)}{\partial dv}\left(dv^{(l+1)} - dv^{(l)}\right) = -h\left(dv^{(l)}\right). \tag{15}$$

Thanks to relation (14), we have

$$\frac{\partial h\left(dv^{(l)}\right)}{\partial dv} = \delta t g'\left(v_k + dv^{(l)}\right) - M \tag{16}$$

so that equation (15) can be written as the following linear system:

$$\begin{aligned}\left[\delta t g'\left(v_k + dv^{(l)}\right) - M\right] dv^{(l+1)} \\ = \left[\delta t g'\left(v_k + dv^{(l)}\right)\right] dv^{(l)} - \delta t g\left(v_k + dv^{(l)}\right)\end{aligned} \tag{17}$$

where $g'(v_k + dv^{(l)}) = (\beta^{-1})'(v_k + dv^{(l)}) f'(\beta^{-1}(v_k + dv^{(l)}))$.

The Newton iterations are started with $dv^{(0)} = 0$ and stopped when $\|h(dv^{(l)})\| < 1e-10$.

IV. Extension to a Semi-Implicit Scheme for the 2-D Vectorial Case

Let us consider an infinitely long superconductor along the z axis. In a two-dimensional setting where the magnetic induction is axial ($\vec{B} = (0,0,B_z)$), the electric field and current density have two nonzero components. The diffusion problem satisfied by the electric field becomes vectorial:

$$\frac{\partial \vec{J}}{\partial t} - \frac{1}{c}\vec{\triangle}\vec{E} = 0. \tag{18}$$

Our idea for getting the solution of this vectorial problem is to solve independently the diffusion equations of each components E_x, E_y of the electric field.

Setting $u_1 = E_x/E_c$, $u_2 = E_y/E_c$ and

$$\beta_1(u_1,u_2) = \left(u_1^2+u_2^2\right)^{\frac{1-n}{2n}} u_1 = J_x/J_c = v_1 \quad (19)$$

$$\beta_2(u_1,u_2) = \left(u_1^2+u_2^2\right)^{\frac{1-n}{2n}} u_2 = J_y/J_c = v_2 \quad (20)$$

the diffusion equations for E_x and E_y are written

$$(S)\begin{cases} \frac{\partial\beta_1(u_1,u_2)}{\partial t} - c^{-1}\triangle u_1 = 0 \text{ in } \Omega \\ \frac{\partial\beta_2(u_1,u_2)}{\partial t} - c^{-1}\triangle u_2 = 0 \text{ in } \Omega \\ \vec{\nabla} u_1\cdot\vec{\nu} = C_{b_1}(t) \text{ on } \partial\Omega \\ \vec{\nabla} u_2\cdot\vec{\nu} = C_{b_2}(t) \text{ on } \partial\Omega \end{cases} \quad (21)$$

where $C_{b_{1,2}}$ are built from Faraday's law and the hypothesis that $\vec{\nabla}\cdot\vec{E} = 0$ on $\partial\Omega$.

We propose a semi-implicit approach in order to handle the nonlinearity of the problem. At each time step, the explicit form u_2^k is used to determine the solution u_1^{k+1} (and vice versa). This assumption leads to the following expression of the nonlinearities:

$$\beta_1\left(u_1^{k+1},u_2^k\right) = \left[\left(u_1^{k+1}\right)^2+\left(u_2^k\right)^2\right]^{\frac{1-n}{2n}} u_1^{k+1} \quad (22)$$

$$\beta_2\left(u_1^{k},u_2^{k+1}\right) = \left[\left(u_1^{k}\right)^2+\left(u_2^{k+1}\right)^2\right]^{\frac{1-n}{2n}} u_2^{k+1}. \quad (23)$$

The Neumann boundary conditions are constructed thanks to this semi-implicit formulation. In order to obtain the fluxes on the boundary, we consider that

$$\vec{\nabla}\cdot\vec{E}=0 \Rightarrow \begin{cases} \frac{\partial u_1^{k+1}}{\partial x} = -\frac{\partial u_2^k}{\partial y}, & \text{unknown is } u_1 \\ \frac{\partial u_2^{k+1}}{\partial y} = -\frac{\partial u_1^k}{\partial x}, & \text{unknown is } u_2 \end{cases} \quad (24)$$

$$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B} \Rightarrow \begin{cases} \frac{\partial u_1^{k+1}}{\partial y} = -\frac{\partial B}{\partial t} - \frac{\partial u_2^k}{\partial x}, & \text{unknown is } u_1 \\ \frac{\partial u_2^{k+1}}{\partial x} = \frac{\partial B}{\partial t} - \frac{\partial u_1^k}{\partial y}, & \text{unknown is } u_1 \end{cases} \quad (25)$$

where $C_{b_{1,2}}$, given in (26) and (27), are deduced from the scalar product with the outward normal vector $\vec{\nu}$

$$C_{b_1}^{k+1} = -\frac{\partial u_2^k}{\partial y}\nu_x - \left(\frac{\partial B}{\partial t} + \frac{\partial u_2^k}{\partial x}\right)\nu_y \quad (26)$$

$$C_{b_2}^{k+1} = \left(\frac{\partial B}{\partial t} - \frac{\partial u_1^k}{\partial y}\right)\nu_x - \frac{\partial u_1^k}{\partial x}\nu_y. \quad (27)$$

The discrete system to solve becomes

$$\begin{cases} M^K \frac{\beta_1\left(u_1^{k+1},u_2^k\right)-\beta_1\left(u_1^k,u_2^k\right)}{\delta t} = f_1\left(u_1^{k+1},u_2^k\right) \\ M^K \frac{\beta_2\left(u_1^{k},u_2^{k+1}\right)-\beta_2\left(u_1^k,u_2^k\right)}{\delta t} = f_2\left(u_1^{k},u_2^{k+1}\right) \end{cases} \quad (28)$$

where f_1 and f_2 are issued from the spatial discretization of the Laplace operator. Unfortunately the inverses of these semi-implicit functions β_1 and β_2 are nontrivial.

In accordance with the $E-J$ power law (2) we suppose that

$$u_1^{k+1} = \beta_1^{-1}\left(v_1^{k+1},v_2^k\right) = \left[\left(v_1^{k+1}\right)^2+\left(v_2^k\right)^2\right]^{\frac{n-1}{2}} v_1^{k+1} \quad (29)$$

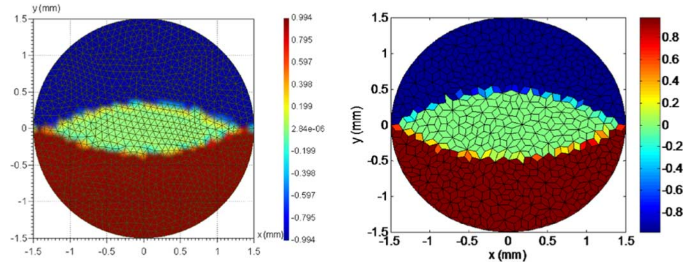

Fig. 2. Current density distribution at $t = 0.5$ s with $n = 200$: (left) DG method with $p = 1$; (right) mixed FE-FV method.

$$u_2^{k+1} = \beta_2^{-1}\left(v_1^{k},v_2^{k+1}\right) = \left[\left(v_1^{k}\right)^2+\left(v_2^{k+1}\right)^2\right]^{\frac{n-1}{2}} v_2^{k+1}. \quad (30)$$

These inverse functions are continuous and differentiable. The previously described Newton iterative method is applied for computing v_1^{k+1} and v_2^{k+1}.

V. Numerical Results

We consider a superconducting cylinder of radius $R = 1.5$ mm, characterized by $J_c = 14.15$ A/mm^2 and $E_c = 10^{-4}$ V/m. The external magnetic flux density field is sinusoidal of amplitude B_0, frequency f and period $T = 1/f$.

A. Scalar Case

When the cylinder is subjected to an external transverse magnetic field in the x direction the diffusion problem of the electric field is scalar and given by (3). We report numerical results obtained with $f = 0.5$ Hz, and $B_0 = 15$ mT.

For $n = 200$, the current density distribution is plotted at $t = T/4$. Notice that we did not get convergence using the H-formulation [2] implemented in the Comsol software. Fig. 2 presents a comparison of our result to that given by the mixed FE-FV method [3]. A good agreement is observed.

B. The 2-D Vectorial Case

Let us now assume that the external magnetic field is applied in the z direction. The diffusion problem vectorial and given by (21). In this example we consider $f = 50$ Hz and $T = 20$ ms. Based on the Bean model, the field required for full penetration will occur at $t = T/4$, where the applied field reaches its maximum value [8].

For $n = 200$, we present the distributions of the components J_x/J_c, J_y/J_c of and the norm $\|\vec{J}\|/J_c$ of the current density at different times. Fig. 3 shows the components in partial penetration at $t = T/8$. They have the expected symmetries because of similarities of their diffusion equations. The penetration form is different compared to the one obtained with a transverse magnetic field. Fig. 4 shows the components in full penetration at $t = T/4$. Symmetries of their distributions are unchanged. The regions where we get iso-values are in a pie chart form.

The modulus $\|\vec{J}\|/J_c = \sqrt{J_x^2/J_c^2 + J_y^2/J_c^2}$ is plotted in Fig. 5. In partial penetration at $t = T/8$, the repartition of the current density is orthoradial. The full penetration of the superconducting cylinder is reached at $t = T/4$ as in the Bean model.

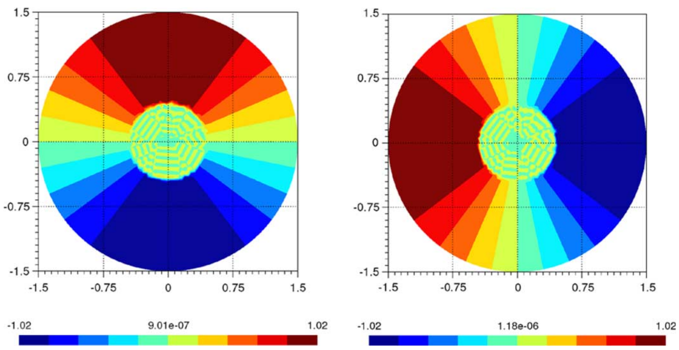

Fig. 3. Current density components at $t = T/8$: (left) J_x/J_c; (right) J_y/J_c.

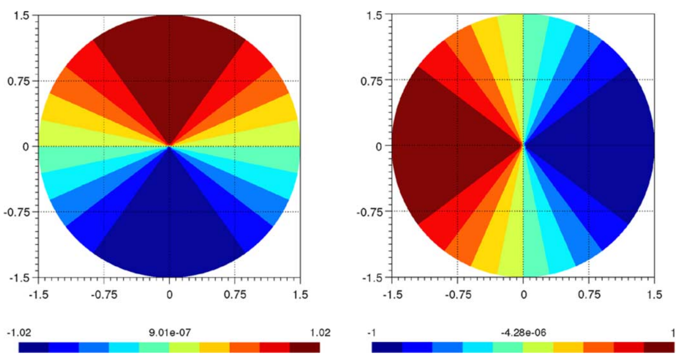

Fig. 4. Current density components at $t = T/4$: (left) J_x/J_c; (right) J_y/J_c.

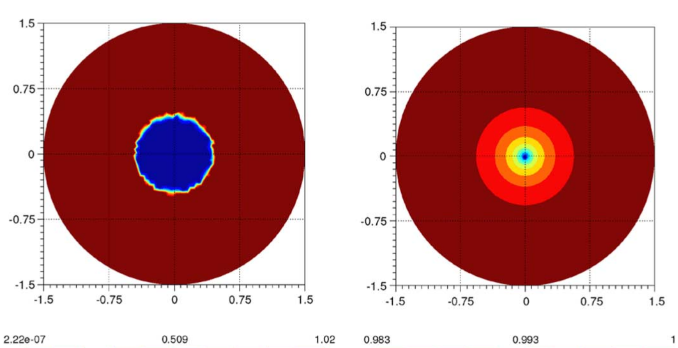

Fig. 5. Norm $J_c^{-1}\sqrt{J_x^2 + J_y^2}$ at different instants: (left) $t = T/8$; (right) $t = T/4$.

C. Example with Nonuniform n

It is well know that in practice the n exponent is not a constant: it locally varies and becomes large in regions where the temperature is close to $0K$.

In this example, we suppose that $n(r) = n_0 \exp(4r/R)$, with $n_0 = 1$ and $r = \sqrt{x^2 + y^2}$. We note that $n(R) = 54$ and $n(0) = 1$. We consider a transverse magnetic field with $B_0 =$ 15 mT and $f = 0.5$ Hz. It is applied in the x-direction, and the diffusion problem of electric field is scalar.

The comparison to the results issued from H-formulation implemented in the Comsol software shows the validity of our approach. Fig. 6 presents the current density distribution at $t =$

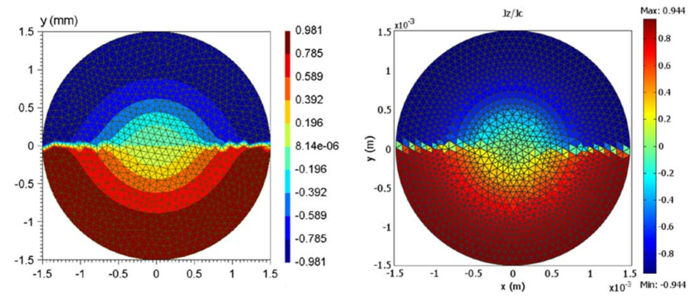

Fig. 6. Current density distribution at $t = 0.5$ s, computed by DG method with $p = 4$ (left) and H-formulation implemented in the Comsol software (right).

$T/4$. Near the border n is large and J/J_c is close to 1. When approaching the center of the cylinder, n decreases and J/J_c becomes smaller.

VI. Conclusion

In this paper, we presented a Discontinuous Galerkin method for solving the nonlinear diffusion problems describing the evolution of the electric field in superconducting materials. The robustness of the approach was highlighted on examples where the exponent in the power constitutive law is large and uniform (for which standard finite-element codes fail to converge), and when the exponent is space-dependent.

The implementation of a semi-implicit $3D$ scheme based on the same approach as the one used in $2D$ is currently in progress.

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