A Modified Fast Hankel Transform Algorithm for Calculating Planar Multilayered Green's Function

P. P. Ding*,† S. Zouhdi* L. W. Li† S. P. Yeo† N. B. Christensen‡

Abstract — **When the Fast Hankel Transform filter technique is used to calculate the dyadic multilayered Green's functions, it can be difficult to obtain accurate numerical results because of the branch-cut singularity and the surface wave poles singularity. The Modified Fast Hankel Transform filter algorithm is proposed to overcome this problem by expressing the Bessel function with a complex argument as a sum of terms of product of Bessel function with the real part of the argument and Bessel function with the imaginary part of the argument. Then the Fast Hankel Transform filter technique is applied to each expansion term. Numerical results confirm that the proposed approach has high accuracy and efficiency and successfully extends the applicability of the conventional Fast Hankel Transform method to general multilayered geometries.**

1 INTRODUCTION

The method of moments (MoM) solution to the integral equation has been widely used for handling multilayered media problems. A crucial computational process for the accurate and efficient MoM analysis is the calculation of the Green's functions for the multilayered media when expressed in terms of Sommerfeld integrals (SI) [1]-[2]. In general, the SIs cannot be analytically evaluated and the numerical integration requires an enormous amount of computational time since the integrand of the integral is highly oscillating and slowly decaying due to Bessel functions and singularities of the spectral domain Green's function. To address this bottleneck, several efficient methods have been proposed to expedite the computation of the SIs . Among those that recently attracted a lot of interests are the discrete complex image method (DCIM) [3], the steepest descent path (SDP) method [4], the window function method (WFM) [5], and the Fast Hankel Transform (FHT) method [6].

The FHT method transforms the SI into a linear discrete convolution and the convolution results can be regarded as the system response of a digital filter. In planar multilayered problems, we often deal with situations where the integrand of SI has SWPs and branch-cut singularities on the integration path. However, since the conventional FHT filters developed until now only permit the input function to be smooth and the branch-cut singularity cannot be removed from the integrand, the conventional FHT method is only applicable for shielded multilayered geometries with the extraction of SWPs.

In this paper, we propose a Modified Fast Hankel Transform (MFHT) filter algorithm to calculate the spatial domain Green's functions for the general multilayered geometries. Firstly, the Sommerfeld integral path (SIP) is deformed from the real axis into the first quadrant to move away from the branch-cut singularity and the SWPs singularity. Secondly, we express the Bessel function with a complex argument as a sum of terms of product of Bessel function with a complex argument and Bessel function with an imaginary argument. For each expansion term, the traditional FHT filter algorithm can be used. To minimize the truncation error and reduce the computational time, the number of expansion terms has to be carefully chosen. Here, we set the maximum relative truncation error to be 10^{-9}, and thus the accuracy of the expansion of Bessel function can be guaranteed.

A detailed presentation of the MFHT filter design is given in Section 2. In Section 3, the accuracy and efficiency of the proposed method are demonstrated by numerical examples. Finally, Section 4 provides the conclusion.

2 MODIFIED FAST HANKEL TRANSFORM METHOD

In the spectral domain Green's function, the branch-cut singularity cannot be analytically removed. Therefore, in order to move away from the SWPs and the branch points to avoid the singularities, the Sommerfeld integration path (SIP) is deformed from the real axis to the first quadrant, as shown in Fig. 1. Then the SI can be written as:

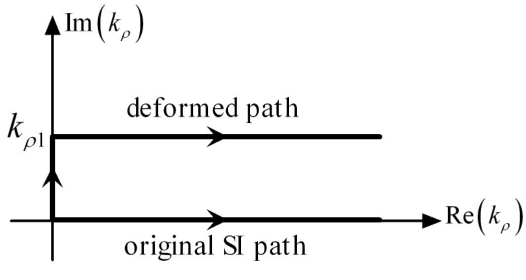

Figure 1: The deformed Sommerfeld integration path.

*Laboratoire de Genie Electrique de Paris, Supelec, Plateau de Moulon, Gif-sur-Yvette, 91192 France.

†Department of Electrical and Computer Engineering, National University of Singapore, 4 Engineering Drive 3, 117583 Singapore.

‡Laboratory of Geophysics, Geological Institute, University of Aarhus, Aarhus, Denmark.

978-1-4244-7368-7/10/$26.00 ©2010 IEEE

$$G(r,r') = \frac{1}{2\pi}\int_0^{jk_{\rho 1}} \widetilde{G}(k_\rho; z, z') J_n(k_\rho \rho) k_\rho^{n+1} dk_\rho + \frac{1}{2\pi}\int_{jk_{\rho 1}}^{\infty + jk_{\rho 1}} \widetilde{G}(k_\rho; z, z') J_n(k_\rho \rho) k_\rho^{n+1} dk_\rho \quad (1)$$

where $k_{\rho 1}$ is a real number. The first integral can be efficiently calculated by adaptive Simpson quadrature method with a computational time quite small compared with the total computational time.

The second integral in (1) can be written as:

$$\begin{aligned} G_2(\rho) &= \frac{1}{2\pi}\int_{jk_{\rho 1}}^{\infty + jk_{\rho 1}} \widetilde{G}(k_\rho; z, z') J_n(k_\rho \rho) k_\rho^{n+1} dk_\rho \\ &= \frac{1}{2\pi}\int_0^{\infty} \widetilde{G}'(k'_\rho; z, z') J_n(k'_\rho \rho + jk_{\rho 1}\rho) dk'_\rho \end{aligned} \quad (2)$$

with

$$k'_\rho = k_\rho - jk_{\rho 1}, \qquad \widetilde{G}'(k'_\rho; z, z') = \widetilde{G}(k_\rho; z, z') \cdot k_\rho^{n+1} \quad (3)$$

where $\rho > 0$ and the input function $\widetilde{G}'$ is a complex function of the real argument k'_ρ. Although the input function of this integral becomes a smooth function along the deformed integration path, the argument of the Bessel function becomes complex. Since the FHT filters developed thus far only permit the argument of Bessel function in the Hankel integral to be real, the traditional FHT method is not directly applicable here.

In order to use the FHT method, a Bessel function with a complex argument can be expressed as [7]:

$$J_n(u \pm v) = \sum_{k=-\infty}^{\infty} J_{n \mp k}(u) J_k(v) \quad (4)$$

The two arguments, u and v, can be arbitrary values and the Bessel functions with complex arguments in (2) are expanded by the sum:

$$J_n(k'_\rho \rho + jk_{\rho 1}\rho) = \sum_{k=-\infty}^{\infty} J_{n-k}(jk_{\rho 1}\rho) J_k(k'_\rho \rho) \quad (5)$$

so that (2) can be written as:

$$\begin{aligned} G_2(\rho) &= \frac{1}{2\pi}\int_0^{\infty} \widetilde{G}'(k'_\rho; z, z') J_n(k'_\rho \rho + jk_{\rho 1}\rho) dk'_\rho \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} J_{n-k}(jk_{\rho 1}\rho) \int_0^{\infty} \widetilde{G}'(k'_\rho; z, z') J_k(k'_\rho \rho) dk'_\rho \end{aligned} \quad (6)$$

where

$$J_m(jk_{\rho 1}\rho) = e^{jm\pi/2} \cdot I_m(k_{\rho 1}\rho) = j^m \cdot I_m(k_{\rho 1}\rho) \quad (7)$$

The modified Bessel function I_m is a monotonic increasing function. Each expansion term in the equation (6) can be evaluated by the traditional FHT method. In this paper, we choose the optimized FHT filter method proposed by [8] to calculate the Hankel integrals since the FHT coefficients for the Hankel transform with an arbitrary order can be easily obtained. The value of $k_{\rho 1}$ and the number of expansion terms are the keys to the accuracy and efficiency of the MFHT algorithm. Here, the value range of $k_{\rho 1}$ is suggested from $0.01k_0$ to $0.026k_0$ and the number of the expansion terms k is from 23 to 29. In the examples of this paper, the number of expansion terms is selected as 27.

In the MFHT filter algorithm, due to the deformed integration path and the optimized FHT method, the time-consuming integration of the product of the slowly decaying input function and oscillating Bessel function is completely avoided.

3 NUMERICAL RESULTS

$$\frac{\rho(1 + jk_0 r)e^{-jk_0 r}}{r^3} = \int_0^{\infty} \frac{e^{-jk_z z}}{jk_z} J_1(k_\rho \rho) {k_\rho}^2 dk_\rho \quad (8)$$

$$\frac{\rho^3(3 + 3jk_0 r - k_0^2 r^2)e^{-jk_0 r}}{r^5} = \int_0^{\infty} \frac{e^{-jk_z z}}{jk_z} J_2(k_\rho \rho) {k_\rho}^3 dk_\rho \quad (9)$$

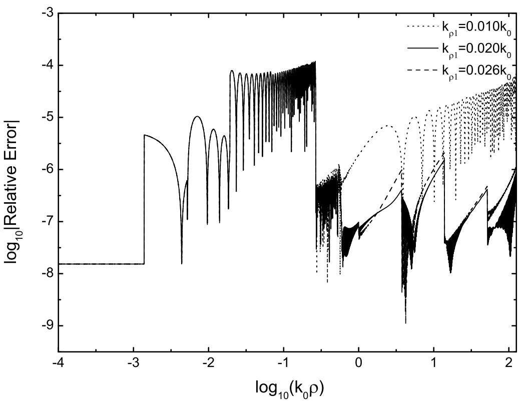

Figure 2: Relative errors of the results calculated by the MFHT filter for the Sommerfeld identity (8), when $z = 1$ mm, $f = 1$ GHz, and $k_{\rho 1} = 0.01k_0$, $0.02k_0$ and $0.026k_0$ respectively.

This section is to investigate the accuracy and efficiency of the MFHT filter algorithm. Firstly, two widely used Sommerfeld identities (8)(9) are used as the numerical examples. The input functions in the two identities have the same singularities located at the points $k_\rho = \pm k_0$. Fig. 2 shows the relative errors for the identity (8) when the operating frequency is 1 GHz and the value of $k_{\rho 1}$ is $0.01k_0$, $0.02k_0$ and $0.026k_0$, respectively. It is seen that accurate results can be obtained when the value range of $k_{\rho 1}$ is from $0.01k_0$ to $0.026k_0$. Fig. 3 depicts the relative errors for the identity (9) when the

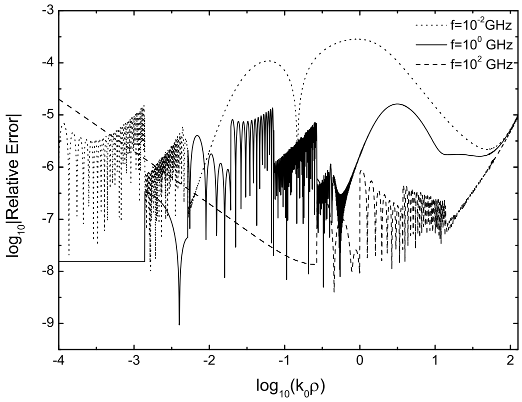

Figure 3: Relative errors of the results calculated by the MFHT filter for the Sommerfeld identity (9), when $z = 1$ mm, $k_{\rho 1} = 0.02k_0$ and the operating frequencies are 10 MHz, 1 GHz and 100 GHz, respectively.

value of $k_{\rho 1}$ is $0.02k_0$ and the operating frequency is 10 MHz, 1 GHz and 100 GHz, respectively. It is illustrated that all the relative errors remain below -2 dB over five times difference of the operating frequency and six times difference of the value of ρ and thus the proposed algorithm can be strongly relied upon when the singularities exist in the input function. All numerical calculations are completed within 5 seconds on a 2.8 GHz personal computer with 1 GB RAM.

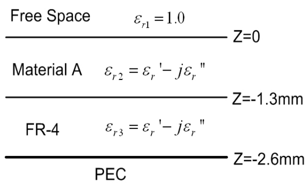

Figure 4: Geometry of two-layer lossy media above a PEC.

Secondly, the MFHT filter algorithm is applied for the calculation of the spatial domain Green's function for the planar multilayered media. For the lossy media shown in Fig. 4, the SWPs are in the fourth quadrant of k_ρ plane and the branch-cut singularity is on the real axis. When $k_{\rho 1} = 0.015k_0$, the smooth input function of the SI is obtained along the deformed SIP, as shown in Fig. 5. The sampling is adequate to capture the characteristic of the spectral domain Green's function $\widetilde{G}_{xx}^{AJ}$. Fig. 6 and Fig. 7 demonstrate the MFHT-based results for G_{xx}^{AJ} and G_{zx}^{AJ}, respectively. The computational results from the MFHT method agree very well with the numerical integration results when the operating frequencies are 0.3 GHz, 3 GHz and 30 GHz, respectively. For each case above, the CPU time used for the MFHT algorithm is within 20 seconds.

It can be inferred from the numerical examples that

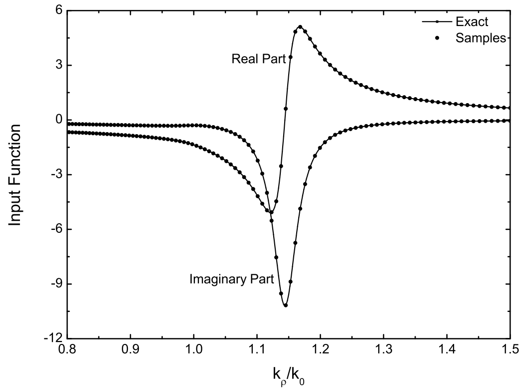

Figure 5: Magnitude of $\widetilde{G}_{xx}^{AJ}$ versus k_ρ/k_0, when $m = 2$, $n = 1$, $z' = -1.3$ mm, $z = 0$ mm, $k_{\rho 1} = 0.015k_0$ and f =30 GHz.

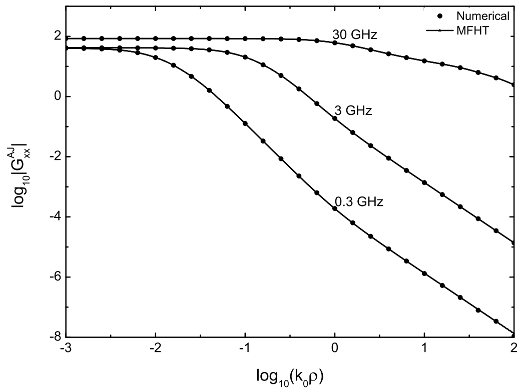

Figure 6: Magnitude Comparison of G_{xx}^{AJ} versus ρ, when $m = 2$, $n = 1$, $z' = -1.3$ mm, $z = 0$ mm, $k_{\rho 1} = 0.015k_0$ and f =0.3, 3, 30 GHz, respectively

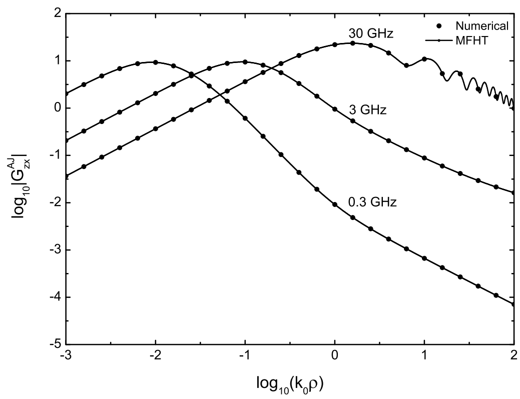

Figure 7: Magnitude Comparison of G_{zx}^{AJ} versus ρ, when $m = 2$, $n = 1$, $z' = -1.3$ mm, $z = 0$ mm, $k_{\rho 1} = 0.015k_0$ and f =0.3, 3, 30 GHz, respectively

the MFHT filter algorithm has excellent accuracy and efficiency when solving the problems with singularities in the input function. When the MFHT method is applied for the calculation of the spatial domain Green's function, the complicated extraction of contribution of the SWPs is completely avoided. Compared with the numerical integration results, the MFHT-based results can be obtained accurately and efficiently.

4 CONCLUSION

In this paper, a Modified Fast Hankel Transform filter algorithm has been proposed to calculate the spatial domain Green's functions for the planar multilayered media. The complicated extraction of contribution of the SWPs singularity can be completely avoided. The rapidly-oscillating and slowly-convergent SIs are transformed into the fast discrete convolutions. Finally, the great accuracy and efficiency of the modified algorithm have been demonstrated when the method is applied for Sommerfeld identities and multilayered problems.

References

[1] K. A. Michalski and D. Zheng, "Electromagnetic scattering and radiation by surface of arbitrary shape in layered media-Part I: Theory," *IEEE Trans. Antennas Propag.*, Vol. 38, 335–344, Mar. 1990.

[2] W. C. Chew, *Wave and Fields in Inhomogeneous Media*. New York: IEEE Press, 1995.

[3] D. G. Fang, J. J. Yang, and G. Y. Delisle, "Discrete Image Theory for Horizontal Electric Dipoles in a Multilayered Medium," *Proc. Inst. Elect. Eng.*, Vol. 135, 297–303, Oct. 1988.

[4] T. J. Cui and W. C. Chew, "Fast Evaluation of Sommerfeld Integrals for EM Scattering and Radiation by Three-Dimensional Buried Objects," *IEEE Trans. Geosci. Remte Sensing.*, Vol. 37, No. 2, 887–900, Mar. 1999.

[5] W. Cai and T.-J. Yu, "Fast Calculations of Dyadic Green's Functions for Electromagnetic Scattering in a Multilayered Medium," *J. Comput. Phys.*, Vol. 165, 1–21, Nov. 2000.

[6] R. C. Hsieh and J. T. Kuo, "Fast Full-Wave Analysis of Planar Microstrip Circuit Elements in Stratified Media," *IEEE Trans. Microw. Theory Tech.*, Vol. 46, No. 9, 1291–1297, Sep. 1998.

[7] M. Abramowitz and I. A. Stegun, *Hankbook of Mathematical Function*. New York: Dover Publications, 1972.

[8] N. B. Christensen, "Optimized fast Hankel transform filters," *Geophys. Prospecing.*, Vol. 38, 545–568, 1990.