A Mortar Edge Element Method with Overlapping for Time Domain Magnetodynamic calculations

A. Christophe*, F. Rapetti†, L. Santandrea*, G. Krebs*, Y. Le Bihan*

**Lab. de Génie Electrique de Paris, UMR 8507 CNRS, SUPELEC, Univ. Paris-Sud, UPMC Univ. Paris 06, France*
alexandra.christophe@lgep.supelec.fr
†Lab. J.-A Dieudonné, UMR 7351 CNRS, Univ. Nice Sophia-Antipolis, France
frapetti@unice.fr

Keywords: Finite element method, edge element approximation, magnetodynamic problem, overlapping meshes, mortar element method.

Abstract

This paper deals with a mortar finite element approach with overlapping grids combined with an edge element discretization in space and an implicit Euler scheme in time. The bidirectional transfer of information between fixed and moving subdomains is described and its accuracy examined. The new method is exposed for a simple magnetodynamic problem.

1 Introduction

The finite element method (FEM) is one of the most used numerical techniques for the modelling of magnetodynamic problems. The FEM has the main advantage of treating problems with various geometries while keeping a simplicity in the implementation. However, some modelisations present the particularity to have nonstationay geometries. A global remeshing can be necessary which causes expensive CPU time. Domain decomposition methods allowing to take into account the movement without having to remesh the whole computational domain have been studied [1, 2]. The mortar element method (MEM) [3], a variational non-conforming domain decomposition approach, offers attractive advantages in terms of flexibility and accuracy. In its original version for non-overlapping subdomains, the idea is to weakly impose the transmission conditions through the skeleton of the decomposition by Lagrangian multipliers [4]. However, it is not always possible to define a coupling interface, which is invariant with movement. Recently, a MEM with overlapping subdomains but unidirectional information transfer has been proposed [5] in order to set free of this constraint. The method of the present work is a new MEM developed to deal with bidirectional information transfert between overlapping subdomains in the frame of edge elements discretization. Concerning the paper organization, the magnetodynamic problem in time domain and the spatial domain decomposition formulation are defined. Then, the overlapping non-matching grid MEM in terms of edge element is described. Finally, a test case is presented and the accuracy of the developed MEM with overlapping is shown.

2 Magnetodynamic formulation

The magnetodynamic problem equations, with $\mathbf{H}$, $\mathbf{B}$, $\mathbf{J}$ and $\mathbf{E}$ denoting the magnetic field, the magnetic flux density, the current density and the electric field, respectively, read:

$$\text{curl}\,\mathbf{H} = \mathbf{J},\ \ \text{curl}\,\mathbf{E} = -\partial_t \mathbf{B},\ \ \text{div}\,\mathbf{B} = 0 \qquad (1).$$

Constitutive laws are considered: $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{J} = \mathbf{J_0} + \sigma\mathbf{E}$ with μ the magnetic permeability, σ the electric conductivity and $\mathbf{J_0}$ the source current density.

Let us consider the domain Ω where a 2D problem (1) is solved. The magnetic vector potential $\mathbf{A}$ is introduced by $\mathbf{B} = \text{curl}\,\mathbf{A}$. In a domain decomposition approach, we separate the fixed $\Omega\backslash\Omega_M$ and the moving Ω_M subdomains in Ω (Fig.1). The considered problem, based on (1) and in term of vector potential $\mathbf{A}$ in $\Omega\backslash\Omega_M$ and $\mathbf{A_M}$ in Ω_M, is reduced to solve the magnetodynamic equations in each subdomain with $(\mathbf{A} - \mathbf{A_M}).\tau_\Gamma = 0$ and $\mu^{-1}\text{curl}\,\mathbf{A} - \mu_M^{-1}\text{curl}\,\mathbf{A_M} = 0$ as interface conditions at $\Gamma = \partial\Omega_M$ with τ_Γ the unit vector tangent of Γ. Dirichlet conditions $\mathbf{A_D}$ are imposed on $\partial\Omega$.

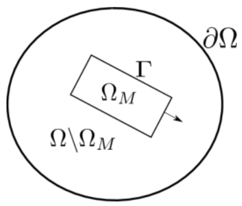

Fig.1: Domain decomposition between a moving subdomain Ω_M and a fixed subdomain Ω.

The associated variational formulation is given by:
Find $(\mathbf{A}, \mathbf{A_M}) \in \text{H}_{\{\mathbf{A_D},\mathbf{A_M}\}}(\text{curl}, \Omega\backslash\Omega_M) \times \text{H}_{\mathbf{A}}(\text{curl}, \Omega_M)$ as:
$\forall \mathbf{A'} \in \text{H}_{\{\mathbf{0},\mathbf{A_M}\}}(\text{curl}, \Omega\backslash\Omega_M), \quad \forall \mathbf{A'_M} \in \text{H}(\text{curl}, \Omega_M)$

$$\partial_t \int_{\Omega\backslash\Omega M} \sigma\mathbf{A}.\mathbf{A'} + \int_{\Omega\backslash\Omega M} \mu^{-1}\text{curl}\,\mathbf{A}\,.\,\text{curl}\,\mathbf{A'} = 0,$$

$$\partial_t \int_{\Omega_M} \sigma\mathbf{A_M}.\mathbf{A'_M} + \int_{\Omega_M} \mu_M^{-1}\text{curl}\,\mathbf{A_M}\,.\text{curl}\,\mathbf{A'_M} = 0, \qquad (2)$$

where $\text{H}_{\{\mathbf{A_D},\mathbf{A_M}\}}(\text{curl}, \Omega\backslash\Omega_M) = \{\mathbf{A} \in \text{H}(\text{curl}, \Omega\backslash\Omega_M), \mathbf{A_M}.\tau_\Gamma = \mathbf{A}.\tau_\Gamma \text{ and } \mathbf{A}.\boldsymbol{\tau}_{\partial\Omega} = \mathbf{A_D}.\boldsymbol{\tau}_{\partial\Omega}\}$ and $\text{H}_{\mathbf{A}}(\text{curl}, \Omega_M) = \{\mathbf{A_M} \in \text{H}(\text{curl}, \Omega_M),\ \mathbf{A_M}.\tau_\Gamma = \mathbf{A}.\tau_\Gamma\}$.

According to the Lax-Milgram theorem, with suitable initial and boundary conditions, equations (2) have a unique solution. The continuous case gives $\mathbf{A}_{|\Gamma} = \mathbf{A}_{\mathbf{M}|\Gamma}$ but the main difficulty arises with the discrete case.

3 Mortar element method

Two discretizations in space depending on each subdomain are considered. They are non-matching grids and completely independent in the overlapping region.

The transfert of information from Ω to Ω_M is realized on Γ by the condition:

$$\mathrm{C}\,\mathbf{A}_{\mathbf{M}|\Gamma} = \mathrm{D}\,\mathbf{A} \tag{3}$$

The reverse transfer from moving to fixed subdomain is carried out through an interface chosen as a boundary (called γ) of the elements of Ω completely overlapped by Ω_M, given as:

$$\mathrm{E}\,\mathbf{A}_{|\gamma} = \mathrm{H}\,\mathbf{A}_{\mathbf{M}} \tag{4}$$

The coupling matrices issued from (3) and (4) are described, on the edges e of the concerned interface, as:

$$\mathrm{C}(i,j) = \int_{e\in\Gamma} (\boldsymbol{w}_i^M.\tau_e)(\boldsymbol{w}_j^M.\tau_e),\ \ \mathrm{D}(i,k) = \int_{e\in\Gamma} (\boldsymbol{w}_i^M.\tau_e)(\boldsymbol{w}_k^F.\tau_e)$$

$$\mathrm{E}(i,j) = \int_{e\in\gamma} (\boldsymbol{w}_i^F.\tau_e)(\boldsymbol{w}_j^F.\tau_e),\ \mathrm{H}(i,k) = \int_{e\in\gamma} (\boldsymbol{w}_i^F.\tau_e)(\boldsymbol{w}_k^M.\tau_e) \tag{5}$$

where $\boldsymbol{w}_i^F$ and $\boldsymbol{w}_i^M$ are the functions of the Whitney edge elements which are defined in the discretizations of Ω and Ω_M, respectively. The matrices C and E concern discrete functions living on the same mesh. The main difficulty is to compute the matrices D and H since both basis functions are defined on different meshes. To solve this problem, the concerned integrals of (5) are divided into the different elements of the triangulations overlapped by the edge e.

The second step in the approximation of problem (2) is its temporal discretization in the interval $[0,T]$ with $T > 0$. A first order implicit Euler scheme is considered.

In order to solve a single algebraic system, the conditions (3) and (4) are strongly imposed in the global matrix by replacing the part of the finite element matrix corresponding to the degrees of freedom of the edges of Γ and γ.

4 Magnetodynamic case

To evaluate the accuracy of the method, a simple magnetodynamic problem with mixed Dirichlet boundary conditions on $\partial\Omega$ are imposed. A disk with a radius equals to 1 m is considered as the global domain Ω. A moving rectangular geometry is used for Ω_M (Fig.2).

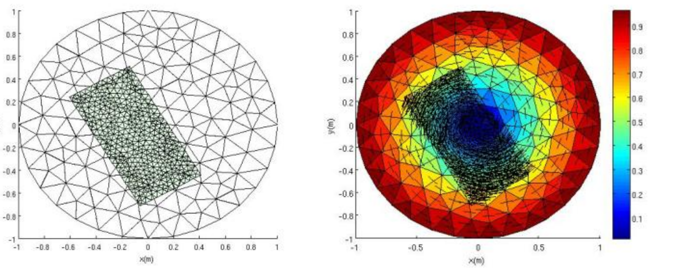

Fig.2:Overlapping meshes: fixed (coarse mesh) and moving (fine mesh) domains (left). Magnetic flux density (in Wb\m^2) and magnetic vector potential (T.m) at the triangle barycenters (right).

The accuracy is maintained with the mortar method. The good distribution of the field is observed between the subdomains(Fig. 2 right). The optimality of the method can be shown in terms of convergence rates from the numerical to the analytical solution(Fig.3).

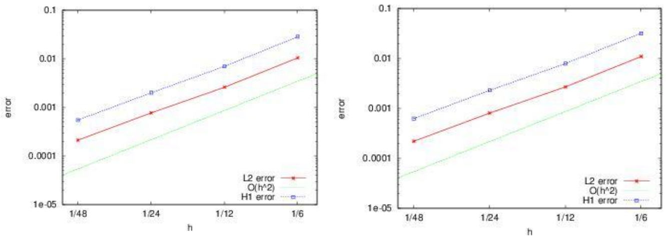

Fig.3: Convergence rates for the solutions in the fixed subdomain (left) and for the global solution (right).

5 Conclusion

This paper presents a new MEM with overlapping subdomains and bidirectional information transfert. This method is introduced for edge elements. In the full paper, the calculation of the coupling matrix will be described, the quadrature formulas detailed and the final linear system defined. Comparisons with an analytical solution and a conventional finite element method without domain decomposition will be presented.

Acknowledgements

This work has the financial support of CEA-LIST.

References

[1] C. R. I. Emson, C. P. Riley, D. A. Walsh, K. Ueda and T. Kumano "Modelling eddy currents induced by rotating systems", *IEEE Trans. Mag.*, vol 34, pp. 2593-2596, (1998).

[2] I. J. Tsukerman. "Overlapping finite element for problems with movement", *IEEE Trans. Mag.*, vol 28, pp. 2247-2249, (1992).

[3] C. Bernardi, Y. Maday, A. Patera. "A new non-conforming approach to domain decomposition: the mortar element method", *Seminaire XI du College de France in Nonlinear partial differential equationsand their applications*, pp. 13-51, (1994).

[4] F.Ben Belgacem. "The Mortar Finite Element Method with Lagrange Multipliers", *Numer. Math.*, vol 84, pp. 173-197, (1999).

[5] F. Rapetti. "An Overlapping Mortar Element Approach to Coupled Magneto-Mechanical Problems", *Math. Comput. Simul.*, vol 80, pp. 1647-1656, (2010).

[6] A. Ben Abdallah, F. Ben Belgacem, Y. Maday, F. Rapetti. "Mortaring the two-dimensional edge finite elements for the discretization of some electromagnetic models", *Mathematical Models and Methods in Applied Sciences*, vol 14 , pp. 1635-1656, (2004).